Int J Theor Phys (2008) 47: 3234–3247
DOI 10.1007/s10773-008-9759-1

Husimi Operator for Describing Probability Distribution of Electron States in Uniform Magnetic Field Studied by Virtue of Entangled State Representation

Qin Guo · Hong-Yi Fan

Received: 8 March 2008 / Accepted: 8 May 2008 / Published online: 20 May 2008

Abstract For the first time we introduce an operator $\Delta_h(\gamma, \varepsilon; \kappa)$ for studying Husimi distribution function in phase space (γ, ε) for electron's states in uniform magnetic field, where κ is the Gaussian spatial width parameter. The marginal distributions of the Husimi function are Gaussian-broadened version of the Wigner marginal distributions. Using the Wigner operator in the entangled state $\langle\lambda|$ representation we find that $\Delta_h(\gamma, \varepsilon; \kappa)$ is just a pure squeezed coherent state density operator $|\gamma, \varepsilon\rangle_{\kappa\kappa}\langle\gamma, \varepsilon|$, which brings much convenience for studying Husimi distribution, so we name $\Delta_h(\gamma, \varepsilon; \kappa)$ the Husimi operator. We then derive Husimi operator's normally ordered form that provides us with an operator version to examine various properties of the Husimi distribution.

Keywords Electron's states · Uniform magnetic field · Husimi distribution function · Entangled state representation · Husimi operator · The technique of integration within ordered product (IWOP)

1 Introduction

Since the discovery of quantum Hall effect [1–10], the motion of an electron in the presence of magnetic field has brought an upsurge of interest. The basic theory that underlies quantum

Work supported by the National Natural Science Foundation under the grant: 10775097.

Q. Guo (✉) · H.-Y. Fan
Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, China
e-mail: guoqin@sjtu.edu.cn

Q. Guo
e-mail: guoqin91@163.com

H.-Y. Fan
e-mail: fhym@sjtu.edu.cn

Q. Guo
Department of Physics, Jiangxi Normal University, Nanchang, Jiangxi 330022, China

Hall effect is the Landau energy-level [11–14]. In [15–17] we have introduced an entangled state representation $|\lambda\rangle$ to describe this system which brings much convenience, for a review we refer to [18]. This coincides with Dirac's guidance in [19]: "When one has a particular problem to work out in quantum mechanics, one can minimize the labor by using a representation in which the representatives of the more important abstract quantities occurring in that problem are as simple as possible". On the other hand, in quantum mechanics it is impossible to specify simultaneously the position Q and the momentum P of a particle due to Heisenberg uncertainty principle. Thus Wigner's quantum phase-space distribution theory [20–32] is of increasing interest because it permits a direct comparison between classical and quantum dynamics. Following the idea of gauge-invariant Wigner operator proposed by Serimaa, Javanainen and Varro [33] we have constructed the corresponding Wigner operator and Wigner function theory for electrons' states in the $|\lambda\rangle$ representation in [34], as well as established the corresponding tomographic theory which means the reconstruction of electron's Wigner distribution from the tomographic data [35].

Let us briefly recall the original idea of Wigner function. Feynman [36] summarized it as posing the following question: If there is any density function $F_w(q,p)$ in quantum mechanics that satisfies

$$\mathrm{P}(p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_w(q,p)dq,\qquad \mathrm{P}(q)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_w(q,p)dp, \tag{1}$$

where $P(q)$ $[P(p)]$ is proportional to the probability for finding the particle at q [at p in momentum space]. The answer is

$$F_w(q,p)=\mathrm{Tr}[\rho\triangle(q,p)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle q+\frac{\upsilon}{2}\right|\rho\left|q-\frac{\upsilon}{2}\right\rangle e^{-ip\upsilon}d\upsilon, \tag{2}$$

where ρ is a density operator, $|q\rangle$ is the eigenvector of the coordinates operator, $Q|q\rangle = q|q\rangle$, and $\triangle(q,p)$ is the single-mode Wigner operator. In the coordinate representation $\triangle(q,p)$ takes the form

$$\begin{aligned}\triangle(q,p)&=\frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty}dud\upsilon\exp[iu(P-p)+i\upsilon(Q-q)]\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|q-\frac{\upsilon}{2}\right\rangle\left\langle q+\frac{\upsilon}{2}\right|e^{-ip\upsilon}d\upsilon,\end{aligned} \tag{3}$$

Equation (1) indicates that $\mathrm{P}(x)[\mathrm{P}(p)]$ is the marginal distribution of $F_w(x,p)$. Using the technique of integration within ordered product (IWOP) of operators [37, 38], we have performed the integral (3) to obtain an explicit operator [39]

$$\triangle(q,p)=\frac{1}{\pi}:e^{-(q-Q)^2-(p-P)^2}:, \tag{4}$$

or

$$\triangle(q,p)\to\triangle(\alpha,\alpha^*)=\frac{1}{\pi}:\exp\left[-2\left(a^\dagger-\alpha^*\right)(a-\alpha)\right]:, \tag{5}$$

where $\alpha=(q+ip)/\sqrt{2}$, : : means normal ordering symbol, $Q=(a+a^\dagger)/\sqrt{2}$, $P=(a-a^\dagger)/(i\sqrt{2})$ is the momentum operator whose eigenvector is $|p\rangle$. It then follows from (4) that one-sided integral over the Wigner operator yields the pure position state density operator

$$\int_{-\infty}^{\infty}dp\triangle(q,p)=\frac{1}{\sqrt{\pi}}:e^{-(q-Q)^2}:=|q\rangle\langle q|, \tag{6}$$

and pure momentum state density operator

$$\int_{-\infty}^{\infty} dq \triangle(q,p) = \frac{1}{\sqrt{\pi}} : e^{-(p-P)^2} := |p\rangle\langle p|, \tag{7}$$

respectively, so the marginal distribution of the Wigner function is $\int_{-\infty}^{\infty} dp \langle\psi|\triangle(q,p)|\psi\rangle = |\psi(q)|^2$ or $\int_{-\infty}^{\infty} dq \langle\psi|\triangle(q,p)|\psi\rangle = |\psi(p)|^2$, respectively. However, as many authors have pointed out that the Wigner function $F_w(q,p)$ is not a probability distribution since it may takes on both positive and negative values. To overcome this shortcomings, the so-called Husimi distribution function $F_h(q,p,\kappa)$ is introduced [40], which is defined in a manner that guarantees it to be non-negative and gives it a probability interpretation. Its definition is smoothing out the Wigner function by averaging over a "coarse graining" function,

$$F_h(q,p,s) = \int\int_{-\infty}^{\infty} dq' dp' F_w\left(q',p'\right) \exp\left[-s\left(q'-q\right)^2 - \frac{(p'-p)^2}{s}\right], \tag{8}$$

where s is the Gaussian spatial width parameter, which determines the relative resolution in p-space versus q-space but is free to be chosen. It is understood that the Husimi density is given by the projection of the wave function ψ onto coherent states localized in phase space (p,q) with a minimum product of the uncertainties $\triangle P = \sqrt{\frac{s\hbar}{2}}$, $\triangle Q = \sqrt{\frac{\hbar}{2s}}$. In this sense s plays the role of squeezing-parameter. In [41, 42] the Husimi operator which corresponds to Husimi function is introduced, which turns out to be a pure squeezed coherent state projector. An interesting question thus naturally arises: how to introduce Husimi functions of phase space for describing probability distribution of electron states in uniform magnetic field (UMF)? To our knowledge, such a question has not been posed in the literature before. As emphasized by Serimaa, Javanainen and Varro [33] that when one wants to establish phase space distribution theory for electron moving in UMF with the gauge potential $\vec{\mathbf{A}} = (-\frac{1}{2}By, \frac{1}{2}Bx, 0)$, electron's canonical momentum operators (p_x, p_y) (conjugate to electron's coordinate operator x, y) should be replaced by its gauge-invariant kinetic momentum (in the units of $\hbar = c = 1$, c denotes the speed of light), $\Pi_x = p_x + eA_x$, $\Pi_y = p_y + eA_y$. Correspondingly, the Wigner operator for describing electrons' motion in UMF should involve Π_x and Π_y as ingredient operators and therefore is gauge invariant. In [34] we have proposed Wigner operator in the entangled state representation (i.e. electron's position representation, denoted by $|\lambda\rangle$). In this work we shall first introduce the Husimi operator $\Delta_h(\varepsilon,\gamma;\kappa)$ by using this Wigner operator. Remarkably, as one can see shortly later, that the Husimi operator $\Delta_h(\varepsilon,\gamma;\kappa)$ is just a pure squeezed coherent state density operator $|\varepsilon,\gamma\rangle_{\kappa\kappa}\langle\varepsilon,\gamma|$, (the explicit form of $|\varepsilon,\gamma\rangle_\kappa$ in Fock space can also be deduced, see (41) below), which brings much convenience to studying Husimi functions for various electron's states. Thus a phase space Husimi distribution theory for electron moving in uniform magnetic field (UMF) can be successfully established.

The work is arranged as follows: In Sect. 2 we briefly review the concise features of the normally ordered form of gauge invariant Wigner operator $\Delta_B(\gamma,\varepsilon)$ in expressing the marginal distribution probability in the $|\lambda\rangle$ representation and its conjugate representation $|\zeta\rangle$ (electron's canonical momentum representation). In Sect. 3 we first introduce the Husimi operator $\Delta_h(\varepsilon,\gamma;\kappa)$ and then derive its normally ordered form, correspondingly, we introduce Husimi function for describing electron's probability distribution. The marginal distributions of Husimi function turns out to be Gaussian-broadened version of the Wigner marginal distributions. We also notice that the Gaussian spatial width parameter can be related to the intensity of magnetic field. In Sect. 4 we introduce the two-mode squeezed coherent state $|\gamma,\varepsilon\rangle_\kappa$ and show its capability of constituting a quantum mechanical representation, we then find that the pure state $|\gamma,\varepsilon\rangle_{\kappa\kappa}\langle\gamma,\varepsilon|$ is just the Husimi operator, so $|\gamma,\varepsilon\rangle_\kappa$ is a good representation for illustrating the Husimi function. In Sect. 5 we further analyze physical explanation of Husimi function of electron's states by calculating the uncertainty relation of electron's position and momentum. In Sect. 6 we calculate the Husimi function of various electron's states in a concise and neat way. In so doing, the Husimi function theory for describing distribution of electron states in uniform magnetic field is established and the relationship between Husimi function and Wigner function is clearly illuminated.

2 Wigner Operator in Entangled State Representation and its Marginal Distributions

The Hamiltonian for electron in UMF is $H=(\Pi_+\Pi_-+\frac{1}{2})\Omega$, the ladder operators are related to electron's kinetic momenta (Π_x,Π_y), $\Pi_\pm=\frac{\Pi_x\pm i\Pi_y}{\sqrt{2M\Omega}}$, $\Omega=\frac{eB}{M}$ is the cyclotron frequency, M is the mass of electron. As the generalization of (3), when the magnetic field applies, the appropriate gauge-invariant Wigner operator is [33]

$$\Delta_B\left(\vec{k},\vec{q}\right)=\frac{1}{(2\pi)^4}\int\int_{-\infty}^{\infty}d^2u d^2v\exp\left[iv\left(\vec{\Pi}-\vec{k}\right)+iu\left(\vec{Q}-\vec{q}\right)\right], \tag{9}$$

where $\vec{k}=(k_1,k_2)$, $\vec{q}=(q_1,q_2)$, $\vec{\Pi}=(\Pi_x,\Pi_y)$, $\vec{Q}=(x,y)$, we have proved in [34] that $\Delta_B(\vec{k},\vec{q})$ in the entangled state representation $|\lambda\rangle$ [15–18] is expressed as (somehow similar in form to (3))

$$\Delta_B(\gamma,\varepsilon)=\int\frac{d^2\lambda}{\pi^3}|\varepsilon^*-\lambda\rangle\langle\varepsilon^*+\lambda|e^{\gamma^*\lambda^*-\gamma\lambda}, \tag{10}$$

where

$$\begin{aligned}&\gamma=\chi+i\sigma^*,\qquad \varepsilon=\chi-i\sigma^*,\\ &\chi=\sqrt{\frac{M\Omega}{2}}(q_1+iq_2)+i\sqrt{\frac{1}{2M\Omega}}(k_1+ik_2),\qquad \sigma=\sqrt{\frac{1}{2M\Omega}}(k_1-ik_2),\end{aligned} \tag{11}$$

the state $|\lambda\rangle$ is

$$|\lambda\rangle=\exp\left[-\frac{1}{2}|\lambda|^2-i\lambda\Pi_++\lambda^*K_++i\Pi_+K_+\right]|00\rangle,\quad \lambda=\lambda_1+i\lambda_2, \tag{12}$$

here the vacuum state is annihilated by $\Pi_-|00\rangle=0$, $K_-|00\rangle=0$, $K_\pm$ are linear combination of guiding centers x_0 and y_0,

$$K_\pm=\sqrt{\frac{M\Omega}{2}}(x_0\mp iy_0), \tag{13}$$

$$x_0=x-\frac{\Pi_y}{M\Omega},\qquad y_0=y+\frac{\Pi_x}{M\Omega}. \tag{14}$$

Note that the above operators obey commutative relations,

$$\begin{aligned}&\left[\Pi_-,\Pi_+\right]=1,\qquad \left[K_-,K_+\right]=1,\\ &\left[K_\pm,\Pi_\pm\right]=0,\qquad \left[x_0,\Pi_\pm\right]=0,\qquad \left[y_0,\Pi_\pm\right]=0,\qquad [x,y]=0,\end{aligned} \tag{15}$$

$$\left[x_0, y_0\right] = \frac{i}{M\Omega}, \qquad \left[\Pi_x, \Pi_y\right] = -iM\Omega,$$

$|\lambda\rangle$ is named entangled state. The motivation of introducing $|\lambda\rangle$ lies in two aspects: Firstly, when magnetic field $\vec{\mathbf{B}}$ applies what we have operators physically describing the system at hand are the guiding centers and kinetic momenta. In other words, the dynamic variables in the Hamiltonian are $\Pi_\pm$, so the corresponding position eigenvector should be expressed by $\Pi_\pm$ as well as $K_\pm$. Secondly, $|\lambda\rangle$ can conveniently describe the position of an electron in a uniform magnetic field, i.e. $|\lambda\rangle$ satisfies the coordinate eigenvector equation

$$(K_+ + i\Pi_-)\,|\lambda\rangle = \lambda|\lambda\rangle, \qquad (K_- - i\Pi_+)\,|\lambda\rangle = \lambda^*|\lambda\rangle. \tag{16}$$

Combining (12)–(16) yields

$$x = \sqrt{\frac{1}{2M\Omega}}(K_+ + K_- - i\Pi_+ + i\Pi_-), \qquad y = \frac{i}{\sqrt{2M\Omega}}(K_+ - K_- + i\Pi_+ + i\Pi_-), \tag{17}$$

$$x\,|\lambda\rangle = \sqrt{\frac{2}{M\Omega}}\lambda_1|\lambda\rangle, \qquad y\,|\lambda\rangle = -\sqrt{\frac{2}{M\Omega}}\lambda_2|\lambda\rangle. \tag{18}$$

Moreover, the Wigner operator expressed by (10) in $|\lambda\rangle$ representation automatically includes the contribution of the magnetic field, this is another merit of introducing $|\lambda\rangle$. The advantage of $\Delta_B(\gamma,\varepsilon)$ also lies in that from (10) we can easily derive its marginal distributions. In fact, using the normally ordered form of $|00\rangle\langle 00| =: \exp[-\Pi_+\Pi_- - K_+K_-]:$ and the IWOP technique [37, 38] we can perform the integration in (10) to derive the normally ordered form of the Wigner operator $\Delta_B(\gamma,\varepsilon)$

$$\begin{aligned}
&\Delta_B(\gamma,\varepsilon) \\
&\quad = \int \frac{d^2\lambda}{\pi^3} :\, \exp\left\{-|\varepsilon^*|^2 - |\lambda|^2 - i\left(\varepsilon^* - \lambda\right)\Pi_+ + \left(\varepsilon - \lambda^*\right)K_+ + i\left(\varepsilon + \lambda^*\right)\Pi_- \right.\\
&\qquad \left. + \left(\varepsilon^* + \lambda\right)K_- + i\Pi_+K_+ - i\Pi_-K_- - \Pi_+\Pi_- - K_+K_- + \gamma^*\lambda^* - \gamma\lambda\right\} : \\
&\quad = \frac{1}{\pi^2} :\, \exp\left\{-\left[\varepsilon^* - (K_+ + i\Pi_-)\right]\left[\varepsilon - (K_- - i\Pi_+)\right]\right.\\
&\qquad \left. - \left[\gamma^* - (K_+ - i\Pi_-)\right]\left[\gamma - (K_- + i\Pi_+)\right]\right\} : .
\end{aligned} \tag{19}$$

As (11) indicates, $\chi = \frac{1}{2}(\gamma+\varepsilon)$, $\sigma^* = \frac{1}{2i}(\gamma-\varepsilon)$, then (19) becomes

$$\Delta_B(\gamma,\varepsilon) = \frac{1}{\pi^2} :\, \exp\left\{-2\left(K_+ - \chi^*\right)(K_- - \chi) - 2\left(\Pi_+ - \sigma^*\right)(\Pi_- - \sigma)\right\} : , \tag{20}$$

which is a 2-dimensional generalization of (5), so (10) is a correct choice. Note that the normally ordered form of the projector $|\lambda\rangle\langle\lambda|$ is

$$|\lambda\rangle\langle\lambda| =:\, \exp\left\{-\left[\lambda^* - (K_- - i\Pi_+)\right]\left[\lambda - (K_+ + i\Pi_-)\right]\right\} : , \tag{21}$$

with the completeness $\int \frac{d^2\lambda}{\pi}|\lambda\rangle\langle\lambda| = 1$, so integrating (19) over $d^2\gamma$ and using (21) we see

$$\pi\langle\psi|\int d^2\gamma\,\Delta_B(\gamma,\varepsilon)|\psi\rangle = :\, \exp\{-[\varepsilon^* - (K_+ + i\Pi_-)][\varepsilon - (K_- - i\Pi_+)]\}:$$

$$= \langle\psi|\lambda\rangle\langle\lambda||_{\lambda=\varepsilon^*}|\psi\rangle = |\langle\psi|\lambda\rangle|^2|_{\lambda=\varepsilon^*}. \tag{22}$$

$|\langle\lambda|\psi\rangle|^2$ is proportional to the probability for finding the electron with position value $[\sqrt{\frac{2}{M\Omega}}\lambda_1, -\sqrt{\frac{2}{M\Omega}}\lambda_2]$. Note $\langle\lambda|\lambda'\rangle = \pi\delta(\lambda-\lambda')\delta(\lambda^*-\lambda'^*) \equiv \pi\delta^{(2)}(\lambda-\lambda')$. On the other hand, integrating (19) over $d^2\varepsilon$ leads to

$$\pi\langle\psi|\int d^2\varepsilon\Delta_B(\gamma,\varepsilon)|\psi\rangle = :\exp\left\{-\left[\gamma^* - (K_+ - i\Pi_-)\right]\left[\gamma - (i\Pi_+ + K_-)\right]\right\}: \tag{23}$$
$$= \langle\psi|\zeta\rangle\langle\zeta||_{\zeta=-\gamma^*}|\psi\rangle = |\langle\psi|\zeta\rangle|^2|_{\zeta=-\gamma^*},$$

where we have defined the state vector $|\zeta\rangle$ as

$$|\zeta\rangle = \exp\left[-\frac{1}{2}|\zeta|^2 - i\zeta\Pi_+ - \zeta^*K_+ - i\Pi_+K_+\right]|00\rangle, \quad \zeta = \zeta_1 + i\zeta_2, \tag{24}$$

and

$$|\zeta\rangle\langle\zeta| = :\exp\{-|\zeta|^2 - i\zeta\Pi_+ - \zeta^*K_+ - i\Pi_+K_+$$
$$+ i\zeta^*\Pi_- - \zeta K_- + i\Pi_-K_- - \Pi_+\Pi_- - K_+K_-\}:$$
$$= :\exp\left\{-\left[\zeta - (i\Pi_- - K_+)\right]\left[\zeta^* - (-i\Pi_+ - K_-)\right]\right\}:, \tag{25}$$

with the completeness $\int\frac{d^2\zeta}{\pi}|\zeta\rangle\langle\zeta| = 1$. $|\zeta\rangle$ is the common eigenvector of the canonical momenta (P_x, P_y), which can be shown as the following. In fact, due to

$$(i\Pi_- - K_+)|\zeta\rangle = \zeta|\zeta\rangle, \qquad (K_- + i\Pi_+)|\zeta\rangle = -\zeta^*|\zeta\rangle \tag{26}$$

and using

$$p_x = \sqrt{\frac{M\Omega}{8}}\left[\Pi_+ + \Pi_- + iK_+ - iK_-\right] = \frac{\Pi_x}{2} + \frac{M\Omega}{2}y_0,$$
$$p_y = \sqrt{\frac{M\Omega}{8}}\left[i\Pi_- - i\Pi_+ - K_+ - K_-\right] = \frac{\Pi_y}{2} - \frac{M\Omega}{2}x_0, \tag{27}$$

we see

$$p_x|\zeta\rangle = \sqrt{\frac{M\Omega}{2}}\zeta_2|\zeta\rangle, \qquad p_y|\zeta\rangle = \sqrt{\frac{M\Omega}{2}}\zeta_1|\zeta\rangle. \tag{28}$$

Thus $|\langle\psi|\zeta\rangle|^2$ in (23) is proportional to the probability for finding the electron with momentum value $(\sqrt{\frac{M\Omega}{2}}\zeta_2, \sqrt{\frac{M\Omega}{2}}\zeta_1)$. Combining (22) and (23) we see that the marginal distributions of the Wigner function for electron states are physical meaningful in the entangled state representation $|\lambda\rangle$ (or $|\zeta\rangle$). This in turn explains that the Wigner operator $\Delta_B(\gamma,\varepsilon)$ expressed in $\langle\lambda|$ representation is a convenient choice which possesses the correct statistical meaning. Note

$$\int d^2\varepsilon\int d^2\gamma\Delta_B(\gamma,\varepsilon) = 1. \tag{29}$$

For a general theory of the entangled Wigner function we refer to [43].

3 Husimi Operator: Normally Ordered Form; The Marginal Distributions of Husimi Distribution Function

In this section we want to introduce the Husimi function $W_h(\gamma,\varepsilon;k)$ for describing electron's probability distribution, the corresponding Husimi operator $\Delta_h(\gamma,\varepsilon;k)$, in reference to (8), is defined as smoothing out $\Delta_B(\gamma',\varepsilon')$ by averaging over a "coarse graining" function,

$$\Delta_h(\gamma,\varepsilon;k)=4\int d^2\gamma' d^2\varepsilon' \Delta_B(\gamma',\varepsilon')\exp\left[-\kappa\left|\varepsilon-\varepsilon'\right|^2-\frac{\left|\gamma-\gamma'\right|^2}{\kappa}\right], \tag{30}$$

where κ is the Gaussian spatial width parameter, which is free to be chosen, and $W_h(\gamma,\varepsilon;k)=\langle\psi|\Delta_h(\gamma,\varepsilon,\kappa)|\psi\rangle$. Using (19) and the IWOP technique we perform the integration in (30),

$$\begin{aligned}
&\Delta_h(\gamma,\varepsilon;k)\\
&=\frac{4}{\pi^2}\int d^2\gamma' d^2\varepsilon' :\exp\{-\left[\varepsilon'^*-(K_+ + i\Pi_-)\right]\left[\varepsilon'-(K_- - i\Pi_+)\right]\\
&\quad-\left[\gamma'^*-(K_+ - i\Pi_-)\right]\left[\gamma'-(K_- + i\Pi_+)\right]\}: \exp\left[-\kappa\left|\varepsilon-\varepsilon'\right|^2-\frac{|\gamma-\gamma'|^2}{\kappa}\right]\\
&=\frac{4\kappa}{(1+\kappa)^2}:\exp\left\{-\frac{\kappa}{1+\kappa}\left[\varepsilon^*-(K_+ + i\Pi_-)\right]\left[\varepsilon-(K_- - i\Pi_+)\right]\right.\\
&\quad\left.-\frac{1}{1+\kappa}\left[\gamma^*-(K_+ - i\Pi_-)\right]\left[\gamma-(K_- + i\Pi_+)\right]\right\}:,
\end{aligned} \tag{31}$$

which is the explicit normally ordered form of the Husimi operator. Using $\gamma=\gamma_1+i\gamma_2$, $\varepsilon=\varepsilon_1+i\varepsilon_2$, (17) and (27) we can further change (31) into the form

$$\begin{aligned}
\Delta_h(\gamma,\varepsilon;k)=\frac{4\kappa}{(1+\kappa)^2}&:\exp\left\{-\frac{\kappa}{1+\kappa}\left[\left(\varepsilon_1-\sqrt{\frac{M\Omega}{2}}x\right)^2+\left(\varepsilon_2-\sqrt{\frac{M\Omega}{2}}y\right)^2\right]\right.\\
&\left.-\frac{1}{1+\kappa}\left[\left(\gamma_1+\sqrt{\frac{2}{M\Omega}}p_y\right)^2+\left(\gamma_2-\sqrt{\frac{2}{M\Omega}}p_x\right)^2\right]\right\}:,
\end{aligned} \tag{32}$$

Using (31) we perform the one-sided integration $d^2\gamma$ over Δ_h,

$$\int\frac{d^2\gamma}{\pi}\Delta_h(\gamma,\varepsilon;k)=\frac{4\kappa}{1+\kappa}:\exp\left\{\frac{-\kappa}{1+\kappa}\left(\varepsilon^*-K_+-i\Pi_-\right)\left(\varepsilon-K_-+i\Pi_+\right)\right\}:. \tag{33}$$

On the other hand, using the $|\lambda\rangle$ representation in (21) and $x|\lambda\rangle=\sqrt{\frac{2}{M\Omega}}\lambda_1|\lambda\rangle$, $y|\lambda\rangle=-\sqrt{\frac{2}{M\Omega}}\lambda_2|\lambda\rangle$ in (18) as well as the IWOP technique we can derive the operator identity

$$\exp\left\{g\left[\left(s_1-\sqrt{\frac{M\Omega}{2}}x\right)^2+\left(s_2-\sqrt{\frac{M\Omega}{2}}y\right)^2\right]\right\}$$

$$= \int \frac{d^2\lambda}{\pi} \exp\{g[(s_1 - \lambda_1)^2 + (s_2 - \lambda_2)^2]\} |\lambda\rangle\langle\lambda|$$

$$= \int \frac{d^2\lambda}{\pi} : \exp\{-(1-g)|\lambda|^2 + \lambda(K_- - i\Pi_+ - gs) + \lambda^*\left(K_+ + i\Pi_- - gs^*\right)$$

$$+ g|s|^2 - (K_- - i\Pi_+)(K_+ + i\Pi_-)\}:$$

$$= \frac{1}{1-g} : \exp\left\{\frac{g}{1-g}\left(s^* - K_+ - i\Pi_-\right)(s - K_- + i\Pi_+)\right\} : , \tag{34}$$

where $s = s_1 + is_2$. So (33) can be simplified as (identifying $-\kappa$ in (33) as g in (34))

$$\int \frac{d^2\gamma}{\pi} \Delta_h(\gamma, \varepsilon; \kappa) = 4\kappa e^{-\kappa\left[\left(\varepsilon_1 - \sqrt{\frac{M\Omega}{2}}x\right)^2 + \left(\varepsilon_2 - \sqrt{\frac{M\Omega}{2}}y\right)^2\right]}, \tag{35}$$

thus the marginal distribution of Husimi operator is a Gaussian operator with the factor κ. It then follows from (35), (22) and (18) the marginal distribution of Husimi function in "λ-direction",

$$\int \frac{d^2\gamma}{\pi} W_h(\gamma, \varepsilon; k) = \langle\psi| \int \frac{d^2\gamma}{\pi} \Delta_h(\gamma, \varepsilon; \kappa) |\psi\rangle$$

$$= 4\kappa \langle\psi| \int \frac{d^2\lambda}{\pi} e^{-\kappa\left[\left(\varepsilon_1 - \sqrt{\frac{M\Omega}{2}}x\right)^2 + \left(\varepsilon_2 - \sqrt{\frac{M\Omega}{2}}y\right)^2\right]} |\lambda\rangle\langle\lambda|\psi\rangle$$

$$= 4\kappa \langle\psi| \int \frac{d^2\lambda}{\pi} e^{-\kappa\left[(\varepsilon_1 - \lambda_1)^2 + (\varepsilon_2 + \lambda_2)^2\right]} |\lambda\rangle\langle\lambda|\psi\rangle$$

$$= 4\kappa \int \frac{d^2\lambda}{\pi} e^{-\kappa|\varepsilon - \lambda^*|^2} |\psi(\lambda)|^2. \tag{36}$$

Comparing (36) with (22) we see that (36) is a Gaussian-broadened version of the quantal position probability distribution $|\psi(\lambda)|^2$ (one marginal distribution of the Wigner function). Similarly, performing the one-sided integration $d^2\varepsilon$ over Δ_h in (32) leads to

$$\int \frac{d^2\varepsilon}{\pi} \Delta_h(\gamma, \varepsilon; \kappa)$$

$$= \frac{4}{1+\kappa} : \exp\left\{-\frac{1}{1+\kappa}\left[\gamma^* - (K_+ - i\Pi_-)\right]\left[\gamma - (i\Pi_+ + K_-)\right]\right\} : . \tag{37}$$

From (25) and (28) as well as the IWOP technique we can prove another operator identity

$$\exp\left\{g\left[\left(v_1 + \sqrt{\frac{2}{M\Omega}}p_y\right)^2 + \left(v_2 - \sqrt{\frac{2}{M\Omega}}p_x\right)^2\right]\right\}$$

$$= \int \frac{d^2\zeta}{\pi} \exp\{g[(v_1 + \zeta_1)^2 + (v_2 - \zeta_2)^2]\} |\zeta\rangle\langle\zeta|$$

$$= \int \frac{d^2\zeta}{\pi} : \exp\{-(1-g)|\zeta|^2 + \zeta(-K_- - i\Pi_+ + gv) + \zeta^*(-K_+ + i\Pi_- + gv^*)$$

$$+ g|v|^2 - (-K_- - i\Pi_+)(-K_+ + i\Pi_-)\}:$$
$$= \frac{1}{1-g} : \exp\left\{\frac{g}{1-g}(v^* - K_+ + i\Pi_-)(v - K_- - i\Pi_+)\right\} : . \tag{38}$$

where $v = v_1 + iv_2$. Thus (37) becomes (identifying $-1/\kappa$ in (37) as g in (38))

$$\int \frac{d^2\varepsilon}{\pi} \Delta_h(\gamma, \varepsilon, \kappa) = \frac{4}{\kappa} e^{-\frac{1}{\kappa}\left[\left(\gamma_1 + \sqrt{\frac{2}{M\Omega}} p_y\right)^2 + \left(\gamma_2 - \sqrt{\frac{2}{M\Omega}} p_x\right)^2\right]}, \tag{39}$$

so the another marginal distribution of (31) is also a Gaussian operator but with the factor $\frac{1}{\kappa}$. It then follows from (39) another marginal distribution of the Husimi function in "ζ-direction"

$$\begin{aligned}
\int \frac{d^2\varepsilon}{\pi} W_h(\gamma, \varepsilon; k) &= \langle\psi| \int \frac{d^2\varepsilon}{\pi} \Delta_h(\gamma, \varepsilon; \kappa) |\psi\rangle \\
&= \frac{4}{\kappa} \langle\psi| \int \frac{d^2\zeta}{\pi} e^{-\frac{1}{\kappa}\left[\left(\gamma_1 + \sqrt{\frac{2}{M\Omega}} p_y\right)^2 + \left(\gamma_2 - \sqrt{\frac{2}{M\Omega}} p_x\right)^2\right]} |\zeta\rangle \langle\zeta\ |\psi\rangle \\
&= \frac{4}{\kappa} \langle\psi| \int \frac{d^2\zeta}{\pi} e^{-\frac{1}{\kappa}\left[(\gamma_1+\zeta_1)^2 + (\gamma_2-\zeta_2)^2\right]} |\zeta\rangle \langle\zeta\ |\psi\rangle \\
&= \frac{4}{\kappa} \int \frac{d^2\zeta}{\pi} e^{-\frac{1}{\kappa}|\gamma^* + \zeta|^2} |\psi(\zeta)|^2,
\end{aligned} \tag{40}$$

which is a Gaussian-broadened version of the quantal momentum probability distribution $|\psi(\zeta)|^2$, (another Wigner marginal distribution (comparing with (23)). Therefore, an operator-representation theory which underlies the Husimi distribution of electron in UMF is established, and the Husimi function's marginal distributions are clear.

4 The Husimi Operator as a Pure Squeezed Coherent State Density Operator

By noticing $|00\rangle\langle 00| =: \exp[-\Pi_+\Pi_- - K_+K_-]:$ we observe that the normally ordered form of the Husimi operator $\Delta_h(\gamma, \varepsilon, \kappa)$ in (31) can be decomposed as

$$\begin{aligned}
&\Delta_h(\gamma, \varepsilon; \kappa) \\
&= \frac{4\kappa}{(1+\kappa)^2} \exp\left\{-\frac{1}{1+\kappa}[\kappa|\varepsilon|^2 + |\gamma|^2 - (\kappa\varepsilon + \gamma) K_+\right. \\
&\quad \left. + i(\kappa\varepsilon^* - \gamma^*)\Pi_+ - i(\kappa - 1)\Pi_+ K_+]\right\} \\
&\quad \times : \exp[-\Pi_+\Pi_- - K_+K_-] : \exp\left\{-\frac{1}{1+\kappa}[-(\kappa\varepsilon^* + \gamma^*)K_-\right. \\
&\quad \left. - i(\kappa\varepsilon - \gamma)\Pi_- + i(\kappa - 1)\Pi_- K_-\right\} \\
&= |\gamma, \varepsilon\rangle_{\kappa\kappa}\langle\gamma, \varepsilon|,
\end{aligned} \tag{41}$$

where we have defined the new state

$$|\gamma,\varepsilon\rangle_\kappa = \frac{2\sqrt{\kappa}}{1+\kappa}\exp\left\{-\frac{1}{1+\kappa}\left[\frac{\kappa|\varepsilon|^2}{2}+\frac{|\gamma|^2}{2}\right.\right.$$
$$\left.\left.-(\kappa\varepsilon+\gamma)K_+ + i\left(\kappa\varepsilon^*-\gamma^*\right)\Pi_+ - i(\kappa-1)\Pi_+K_+\right]\right\}|00\rangle. \tag{42}$$

Thus the Husimi operator $\Delta_h(\lambda,\zeta,\kappa)$ is just the pure state density operator $|\gamma,\varepsilon\rangle_{\kappa\kappa}\langle\gamma,\varepsilon|$, this is a remarkable result. It turns out that $|\gamma,\varepsilon\rangle_\kappa$ is a two-mode squeezed canonical coherent state because it obeys the eigenvector equations

$$(K_-\cosh r + i\Pi_+\sinh r)|\gamma,\varepsilon\rangle_\kappa = \frac{\sqrt{\kappa}\varepsilon+\gamma/\kappa}{2}|\gamma,\varepsilon\rangle_\kappa \tag{43}$$

and

$$(\Pi_-\cosh r + iK_+\sinh r)|\gamma,\varepsilon\rangle_\kappa = i\frac{\gamma^*/\sqrt{\kappa}-\sqrt{\kappa}\varepsilon^*}{2}|\gamma,\varepsilon\rangle_\kappa \tag{44}$$

where $\frac{1-\kappa}{1+\kappa}\equiv\tanh r$ is a squeezing parameter, $e^r=\frac{1}{\sqrt{\kappa}}$, $\cosh r=\frac{1+\kappa}{2\sqrt{\kappa}}$. The corresponding squeezing operator is

$$S(r)\equiv e^{i(xp_x+yp_y-i)r} = \exp\left[ir(\Pi_+K_+ + \Pi_-K_-)\right], \tag{45}$$

(For reviews of general squeezed state theory in quantum optics we refer to [44–51]). The disentangling of (45) is

$$S(r) = \operatorname{sech} r\exp(i\Pi_+K_+\tanh r)\exp[(K_+K_- + \Pi_+\Pi_-)\ln\operatorname{sech} r]$$
$$\times\exp(i\Pi_-K_-\tanh r). \tag{46}$$

From (46), (14)–(15) we derive

$$\begin{aligned} S^{-1}K_-S &= K_-\cosh r + i\Pi_+\sinh r, & S^{-1}\Pi_-S &= \Pi_-\cosh r + iK_+\sinh r,\\ S^{-1}K_+S &= K_+\cosh r - i\Pi_-\sinh r, & S^{-1}\Pi_+S &= \Pi_+\cosh r - iK_-\sinh r, \end{aligned} \tag{47}$$

and using (18) and (27) we have

$$S^{-1}xS = \sqrt{\kappa}x, \qquad S^{-1}yS = \sqrt{\kappa}y, \tag{48}$$

$$S^{-1}p_xS = p_x/\sqrt{\kappa}, \qquad S^{-1}p_yS = p_y/\sqrt{\kappa}. \tag{49}$$

In (19) we see that λ denotes the eigenvalue of electron's coordinates, so $S(r)$ has a natural representation in $\langle\lambda|$ representation [15–17]

$$S(r) = e^{-r}\int\frac{d^2\lambda}{\pi}\left|e^{-r}\lambda\right\rangle\langle\lambda|, \qquad e^r = \frac{1}{\sqrt{\kappa}}, \tag{50}$$

from $\langle\lambda|\lambda'\rangle = \pi\delta^{(2)}(\lambda-\lambda')$, $S(r)|\lambda\rangle = e^{-r}|e^{-r}\lambda\rangle$, so (50) embodies another merit of constructing the entangled state representation $|\lambda\rangle$. From the eigenvalue equations (19) we also see that the eigenvalue of x and y varies with B, since $\sqrt{\frac{1}{M\Omega}} = \frac{1}{\sqrt{eB}}$, so the variation of the magnetic field intensity B is related to squeezing of electron's orbit track. Thus the variation of Gaussian spatial width parameter $\sqrt{\kappa}$ can also be interpreted as the change of magnetic field intensity $\sqrt{B}$. From (43)–(44) we notice that $|\gamma,\varepsilon\rangle_\kappa$ can be expressed as the result of the squeezing operator operating on the state $|\gamma,\varepsilon\rangle$, i.e.

$$|\gamma,\varepsilon\rangle_\kappa = S^{-1}(r)\,|\gamma,\varepsilon\rangle, \tag{51}$$

where

$$\begin{aligned}|\gamma,\varepsilon\rangle \equiv \exp\Bigg[&-\frac{1}{4}\left(\kappa|\varepsilon|^2+|\gamma|^2/\kappa\right)+i\,\frac{\gamma^*/\sqrt{\kappa}-\sqrt{\kappa}\varepsilon^*}{2}\Pi_+ \\ &+\frac{\sqrt{\kappa}\varepsilon+\gamma/\sqrt{\kappa}}{2}K_+\Bigg]|00\rangle,\end{aligned} \tag{52}$$

is a normalized two-mode coherent state for an electron in UMF, and we have dropped the inconsequential phase factor $\exp\{\frac{\kappa-1}{4(1+\kappa)}(\varepsilon^*\gamma-\gamma^*\varepsilon)\}$ in the result of calculating $S^{-1}(r)|\gamma,\varepsilon\rangle$.

5 Further Explanation of the Husimi Function

Using (52), (48) and (18) we see that in the state $|\gamma=0,\varepsilon=0\rangle_\kappa$ the variance of electron's position x is

$$\begin{aligned}(\Delta x)^2 &\equiv {}_\kappa\langle 0,0|\,x^2\,|0,0\rangle_\kappa-({}_\kappa\langle 0,0|\,x\,|0,0\rangle_\kappa)^2 = \langle 0,0|\,S(r)\,x^2S^{-1}(r)\,|00\rangle \\ &= \frac{1}{2M\Omega\kappa}\,\langle 00|\,(K_++K_--i\Pi_++i\Pi_-)^2\,|00\rangle=\frac{1}{M\Omega\kappa},\end{aligned} \tag{53}$$

while the variances of p_x is

$$\begin{aligned}(\Delta p_x)^2 &= \langle 00|\,S(r)\,p_x^2S^{-1}(r)\,|00\rangle \\ &= \frac{\kappa M\Omega}{8}\langle 0,0|[\Pi_++\Pi_--iK_++iK_-]^2\,|00\rangle=\frac{\kappa M\Omega}{4}.\end{aligned} \tag{54}$$

On the other hand, $|\gamma,\varepsilon\rangle_\kappa$ is complete

$$\frac{1}{4\pi^2}\int d^2\varepsilon\int d^2\gamma\,|\gamma,\varepsilon\rangle_{\kappa\kappa}\langle\gamma,\varepsilon|=1, \tag{55}$$

so the Husimi density

$$\langle\psi|\,\Delta_h(\gamma,\varepsilon,\kappa)\,|\psi\rangle=\left|\langle\psi\,|\gamma,\varepsilon\rangle_\kappa\right|^2 \tag{56}$$

is given by the projection of the wave function onto the squeezed coherent states localized in phase space with a minimum product of the uncertainties

$$\Delta p_x=\sqrt{\frac{\kappa M\Omega}{4}},\qquad \Delta x=\sqrt{\frac{1}{M\Omega\kappa}},\qquad \Delta x\,\Delta p_x=\frac{1}{2}. \tag{57}$$

In this sense the Gaussian spatial width parameter $\kappa=\frac{2\Delta p_x}{M\Omega\Delta x}=\frac{2\Delta p_x}{eB\Delta x}$ plays the role of squeezing-parameter (note that in the units of $\hbar=c=1$, $\sqrt{\frac{2}{eB}}$ is the magnetic length.)

Further, using (41) we can re-express the marginal distribution (40) of the Husimi function of electron's quantum state $|\psi\rangle$ as

$$\int \frac{d^2\varepsilon}{\pi} W_h(\gamma, \varepsilon; k) = \int \frac{d^2\varepsilon}{\pi} |_{\kappa} \langle \gamma, \varepsilon | \psi \rangle|^2. \tag{58}$$

We can also recast (36) as

$$\int \frac{d^2\gamma}{\pi} W_h(\gamma, \varepsilon; \kappa) = \int \frac{d^2\gamma}{\pi} |_{\kappa} \langle \gamma, \varepsilon | \psi \rangle|^2. \tag{59}$$

Equations (58) and (59) indicate the relationship between probability density of $|\psi\rangle$ in the $_{\kappa}\langle \gamma, \varepsilon|$ representation and those in the entangled state $\langle \lambda|$ representation.

6 Husimi Functions of Some Electron's States

Using (42) we can derive the Husimi function of $|\varepsilon', \gamma'\rangle_k$,

$$\begin{aligned} {}_{\kappa}\left\langle \varepsilon', \gamma' \right| \Delta_h(\gamma, \varepsilon; \kappa) \left| \varepsilon', \gamma' \right\rangle_{\kappa} &= \left|_{\kappa} \langle \gamma, \varepsilon | \gamma', \varepsilon' \right\rangle_{\kappa}\right|^2 \\ &= \exp\left[-\frac{\kappa}{2} \left| \varepsilon' - \varepsilon \right|^2 - \frac{|\gamma' - \gamma|^2}{2\kappa} \right], \end{aligned} \tag{60}$$

which is a Gaussian broadened function, and the Husimi function of the electron's coordinate eigenstate $|\lambda\rangle$,

$$\langle \lambda | \Delta_h(\gamma, \varepsilon; \kappa) | \lambda \rangle = \kappa |\langle \sqrt{\kappa} \lambda | \gamma, \varepsilon \rangle|^2 = \kappa \exp\{-\kappa |\lambda - \varepsilon^*|^2\}, \tag{61}$$

which is also a Gaussian. This is in sharply contrast with the Wigner function of $|\lambda\rangle$ which can be calculated by using (11)

$$\begin{aligned} \langle \lambda | \Delta_B(\gamma, \varepsilon) | \lambda \rangle &\equiv \langle \lambda | \int \frac{d^2\lambda'}{\pi^3} \left| \varepsilon^* - \lambda' \right\rangle \left\langle \varepsilon^* + \lambda' \right| e^{\gamma^* \lambda'^* - \gamma \lambda'} | \lambda \rangle \\ &= \int \frac{d^2\lambda'}{\pi} \delta^{(2)}\left(\lambda - \varepsilon^* + \lambda' \right) \delta^{(2)}\left(\lambda - \varepsilon^* - \lambda' \right) e^{\gamma^* \lambda'^* - \gamma \lambda'} \\ &= \frac{1}{4\pi} \delta^{(2)}(\lambda - \varepsilon^*). \end{aligned} \tag{62}$$

Comparing (61) and (62) and recall the limiting Gaussian-form of Delta function we can see again that Husimi function is the Gaussian-broadened version of Wigner function. Next we calculate Landau state's Husimi function,

$$\begin{aligned} \langle n, m | \Delta_h(\gamma, \varepsilon; \kappa) | n, m \rangle &= |\langle n, m | \gamma, \varepsilon \rangle_{\kappa}|^2 \\ &= \frac{1}{n!m!} \frac{4\kappa}{(1+\kappa)^2} \left(\frac{1-\kappa}{1+\kappa} \right)^{n+m} \exp\left(-\frac{\kappa |\varepsilon|^2 + |\gamma|^2}{1+\kappa} \right) \\ &\quad \times \left| H_{m,n}\left(\frac{-(\kappa\varepsilon + \gamma)}{\sqrt{\kappa^2 - 1}}, \frac{-(\kappa\varepsilon^* - \gamma^*)}{\sqrt{\kappa^2 - 1}} \right) \right|^2. \end{aligned} \tag{63}$$

where $H_{m,n}$ is two-variable Hermite polynomial [52] whose definition is

$$H_{m,n}(x,y) = \sum_{l=0}^{\min(m,n)} \frac{m!n!(-1)^l}{l!(m-l)!(n-l)!} x^{m-l} y^{n-l}. \tag{64}$$

In summary, for the first time we have introduced the Husimi operator $\Delta_h(\gamma, \varepsilon; \kappa)$ for electron in UMF, and shown $\Delta_h(\gamma, \varepsilon; \kappa) = |\gamma, \varepsilon\rangle_{\kappa\kappa} \langle\gamma, \varepsilon|$, i.e. the Husimi operator actually is a pure squeezed coherent state projector. The normally ordered form of Husimi operator are also derived which provides us with an operator version to examine various properties of the Husimi distribution. We have in many ways demonstrated that Husimi (marginal) distributions are Gaussian-broadened version of the Wigner (marginal) distributions. Throughout the paper we have fully employed the technique of integration within an ordered product of operators and the entangled state representation, each of them seems an efficient method for studying quantum statistical physics [53].

References

1. Klitzing, K.V., Dorda, G., Pepper, M.: Phys. Rev. Lett. **45**, 494 (1980)
2. Tsui, D.C., Stormer, H.L., Gossard, A.C.: Phys. Rev. Lett. **48**, 1559 (1982)
3. Laughlin, R.B.: Phys. Rev. Lett. **50**, 1395 (1983)
4. Laughlin, R.B.: Phys. Rev. Lett. **50**, 1395 (1983); ibid. **60**, 1057 (1988)
5. Laughlin, R.B.: Science **242**, 525 (1988)
6. Laughlin, R.B.: Rev. Mod. Phys. **71**, 863 (1999)
7. Laughlin, R.B.: Phys. Rev. B **23**, 5632 (1981)
8. Girvin, S.M., Jach, T.: Phys. Rev. B **28**, 4506 (1983)
9. Haldane, F.D.M.: In: Prange, R.E., Girvin, S.M. (eds.) The Quantum Hall Effect. Springer, Berlin (1987)
10. Charkraborty, T., Pietiläinen, P.: The Quantum Hall Effects (Fractional and Integral), 2nd edn. Springer, Berlin (1995)
11. Landau, L.D.: Z. Phys. **64**, 629 (1930)
12. Landau, L.D., Lifshitz, E.M.: Quantum Mechanics. Pergamon, Elmsford (1977)
13. Johnson, M.H., Lippmann, B.A.: Phys. Rev. **76**, 828 (1946)
14. Feldman, A., Kahn, A.H.: Phys. Rev. B **1**, 4584 (1970)
15. Fan, H.-Y.: Phys. Lett. A **126**, 150 (1987)
16. Fan, H.-Y., Ren, Y.: Mod. Phys. Lett. B **10**, 523 (1996)
17. Fan, H.-Y., Zhang, Y.: Mod. Phys. Lett. B **11**, 239 (1997)
18. Fan, H.-Y.: Int. J. Mod. Phys. B **18**, 2771–2817 (2004) (For a review)
19. Dirac, P.A.M.: The Principle of Quantum Mechanics, 4th edn. Oxford University Press, London (1958)
20. Wigner, E.: Phys. Rev. **40**, 749 (1932)
21. Weyl, H.: Z. Phys. **46**, 1 (1927)
22. O'Connell, R.F., Wigner, E.P.: Phys. Lett. A **83**, 145 (1981)
23. Hillery, M., O'Connell, R.F., Scully, M.O., Wigner, E.P.: Phys. Rep. **106**, 121–167 (1984)
24. Vogel, K., Risken, H.: Phys. Rev. A **40**, 2847 (1989)
25. Smithey, D.T., Beck, M., Raymer, M.G.: Phys. Rev. Lett. **70**, 1244 (1993)
26. Wunsche, A.: J. Mod. Opt. **44**, 2293 (1997)
27. Wunsche, A.: J. Mod. Opt. **33**, 47 (2000)
28. Guo, Q.: Int. J. Theor. Phys. **46**, 3135 (2007)
29. Agawal, G.S., Wolf, E.: Phys. Rev. D **2**, 2161 2187 2206 (1972)
30. Bužek, V., Knight, P.L.: Prog. Opt. **34**, 1–158 (1995)
31. Bužek, V., Keitel, C.H., Knight, P.L.: Phys. Rev. A **51**, 2575 (1995)
32. Dodonov, V.V., Man'ko, V.I.: Theory of Nonclassical States of Light. Taylor & Francis, New York (2003)
33. Serimaa, O.T., Javanainen, J., Varro, S.: Phys. Rev. A **33**, 2913 (1986)
34. Fan, H.-Y., Yang, Z.-S.: Mod. Phys. Lett. A **13**, 2679 (1998)

35. Fan, H.-Y.: Phys. Lett. A **301**, 153 (2002)
36. Feynman, R.P.: Statistical Physics. W.A. Benjamin, Elmsford (1972)
37. Fan, H.-Y.: J. Opt. Phys. B: Quantum Semiclass. Opt. **5**, R147–163 (2003)
38. Wünsche, A.: J. Opt. B: Quantum Semiclass. Opt. **1**, R11–21 (1999)
39. Fan, H.-Y., Zaidi, H.R.: Phys. Lett. A **124**, 303 (1987)
40. Husimi, K.: Proc. Phys. Math. Soc. Jpn. **22**, 264 (1940)
41. Fan, H.-Y., Yang, Y.-L., Phys. Lett. A **353**, 439 (2006)
42. Fan, H.-Y., Guo, Q.: Phys. Lett. A **358**, 203 (2006)
43. Fan, H.-Y.: Phys. Rev. A **65**, 064102 (2002)
44. D'Ariano, G.M., Rassetti, M.G., Katriel, J., Solomon, A.I.: In: Tombesi, P., Pike, E.R. (eds.) Squeezed and Nonclassical Light, p. 301. Plenum, New York (1989)
45. Bužek, V.: J. Mod. Opt. **37**, 303 (1990)
46. Loudon, R., Knight, P.L.: J. Mod. Opt. **34**, 709 (1987)
47. Orszag, M.: Quantum Optics. Springer, Berlin (2000)
48. Schleich, W.P.: Quantum Optics in Phase Space. Wiley, Berlin (2001)
49. Scully, M.O., Zubairy, M.S.: Quantum Optics. Cambridge University Press, Cambridge (1997)
50. Dodonov, V.V.: J. Opt. B: Quantum Semiclass. Opt. **4**, R1–33 (2002)
51. Walls, D.F., Milburn, G.J.: Quantum Optics. Springer, Berlin (1994)
52. Erdelyi, A.: Higher Transcendental Functions (The Bateman Manuscript Project). McGraw-Hill, New York (1953)
53. Fan, H.-Y., et al.: Ann. Phys. **321**, 480 (2006)